Universität Konstanz
Fachbereich Mathematik und Statistik
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Exercises for the course "Linear Algebra I"

Sheet 10

Hand in your solutions on Thursday, 16. January 2020, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 10.1 (*5 points*)

Let K be a field, $n \in \mathbb{N}$ and $A, B \in M_{n\times n}(K)$ such that $AB = I_n$. We also say that A is a *left inverse* of B.

(a) Let $X \in K^n$. Show that $BX = 0$ if and only if $X = 0$. Deduce that the system of linear equations $BX = C$ admits a solution for all $C \in K^n$.

(b) Show that A is also a *right inverse* of B i.e., $BA = I_n$.

Now let $A, B \in M_{n\times n}(K)$ be arbitrary matrices.

(c) Assume $AB + A + B = 0$. Show that $AB = BA$.

Exercise 10.2 (*4 points*)

Let K be a field and let U and V be finite dimensional K-vector spaces.

(a) Give a basis and the dimension of the K-vector space $U \times V$ and justify your answer.

(b) Let W be a K-vector space containing U and V. Use the dimension theorem (Satz 18.2) to prove that

$$\dim U + \dim V = \dim(U + V) + \dim(U \cap V).$$

Exercise 10.3 (*4 points*)

Let K be a field, V and W finite dimensional K-vector spaces and $T\colon V \to W$ a linear map. Moreover, let $\mathcal{B}_{\text{ker}} = \{v_1, \ldots, v_n\}$ be a basis of the kernel of T and $\mathcal{B}_{\text{im}} = \{T(u_1), \ldots, T(u_m)\}$ a basis of the image of T, where $n, m \in \mathbb{N}_0$ and $v_i, u_j \in V$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$.

Show that $\mathcal{B} := \{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ is a basis of V.

Exercise 10.4 (*3 points*)

It follows from the dimension theorem that a linear map between two n-dimensional vector spaces is injective if and only if it is surjective (this is shown in class). Here you will show that this is not the case for infinite dimensional vector spaces. For that purpose consider the following map:

$$\varphi\colon K[X] \longrightarrow K[X], \quad f = \sum_{i=0}^{n} a_i X^i \longmapsto f' := \sum_{i=0}^{n-1} (i+1) a_{i+1} X^i$$

for an arbitrary field K of characteristic 0. Prove that φ is a linear map, that it is surjective and that it is **not** injective.

Further show that φ is not surjective for any field K of non-zero characteristic.